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PACKING NEARLY-DISJOINT SETS

P. D. SEYMOUR

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De Bruijn and Erdös proved that if $A_1, \ldots, A_k$ are distinct subsets of a set of cardinality n, and $|A_i \cap A_j| \leqq 1$ for $1 \leqq i < j \leqq k$, and $k > n$, then some two of $A_1, \ldots, A_k$ have empty intersection. We prove a strengthening, that at least $\frac{k}{n}$ of $A_1, \ldots, A_k$ are pairwise disjoint. This is motivated by a well-known conjecture of Erdős, Faber and Lovász of which it is a corollary.

1. Introduction

Throughout this paper we shall only be concerned with finite structures, and further reference to this will be omitted.

If $\mathscr{A}$ is a collection of sets, we say that $\mathscr{A}$ is *nearly-disjoint* if $|A \cap A'| \leqq 1$ for distinct $A, A' \in \mathscr{A}$. We begin by describing a conjecture of Erdős, Faber and Lovász (see for example [2]), which motivated this paper. It is the following.

(1.1) (Conjecture). *Let $n > 0$ be an integer, and let $\mathscr{A}$ be a nearly-disjoint collection of sets, each with cardinality n, and let $|\mathscr{A}| = n$. Then the elements of $\bigcup(A\colon A \in \mathscr{A})$ may be coloured with n colours, so that for each $A \in \mathscr{A}$, all the elements of A receive different colours.*

It may be shown to be equivalent to the following.

(1.2) (Conjecture). *Let $\mathscr{A}$ be a nearly-disjoint collection of subsets of a set of cardinality $n > 0$. Then the members of $\mathscr{A}$ may be coloured with n colours, so that distinct $A, A \in \mathscr{A}$ receive different colours if $A \cap A' \neq \emptyset$.*

We prove this equivalence in section 2. Our main result is the following.

(1.3) *Let $\mathscr{A}$ be a nearly-disjoint collection of subsets of a set of cardinality $n > 0$. Then there exist $A_1, \ldots, A_p \in \mathscr{A}$, pairwise disjoint, where $p \geqq \frac{1}{n} |\mathscr{A}|$.*

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Clearly this is a corollary of (1.2) since some colour class must contain at least $\frac{1}{n}|\mathscr{A}|$ members of $\mathscr{A}$. We prove (1.3) in section 3, and in section 4 we find the collections for which 1.3 is best possible. Section 5 contains some concluding remarks.

Let us mention in passing that (1.3) contains a well-known result of De Bruijn and Erdős [1], the following.

(1.4) *Let $\mathscr{A}$ be a collection of subsets of a set of cardinality $n>0$, such that $|A\cap A'|=1$ for distinct $A, A'\in\mathscr{A}$. Then $|\mathscr{A}|\leqq n$.*

2. The Equivalence of (1.1) and (1.2)

Let $\mathscr{A}$ be a nearly-disjoint collection of subsets of a set E. (Note that "collection" means "set", and a collection has no repeated members.) Consider the following statements about $\mathscr{A}$.

(2.1) $|\mathscr{A}|=n>0$.
(2.2) $|A|=n$ *for each* $A\in\mathscr{A}$.
(2.3) *The elements of E may be coloured with n colours so that for each $A\in\mathscr{A}$, all the elements of A receive different colours.*

Thus conjecture (1.1) asserts that **(2.1)** and (2.2) imply (2.3). Now consider the statement
(2.4) $|A|\leqq n$ *for each* $A\in\mathscr{A}$.

(1.1) is equivalent to the assertion that (2.1) and (2.4) imply (2.3), for if some $A\in\mathscr{A}$ has $|A|<n$, we simply add $n-|A|$ new elements to A. Finally, consider the following statement.

(2.5) *For distinct $x, x'\in E$, there exists $A\in\mathscr{A}$ such that $|A\cap\{x, x'\}|=1$.*

We may add (2.5) to our list of hypotheses; for if (2.1), (2.4) and (2.5) together imply (2.3), then (2.1) and (2.4) imply (2.3), as is easily seen. But (2.4) is implied by (2.1) and (2.5). Hence (1.1) is equivalent to the assertion that (2.1) and (2.5) imply (2.3). However, (2.5) ensures that the set-element dual of $\mathscr{A}$ is another collection of sets, and it is clearly nearly-disjoint. (1.2) is merely the assertion that (2.1) and (2.5) imply (2.3), expressed in terms of the set-element dual of $\mathscr{A}$. Thus (1.1) and (1.2) are equivalent.

3. Proof of the theorem 1.3

Let p be the least integer with $p\geqq\frac{1}{n}|\mathscr{A}|$. We proceed by induction on p. The result is trivial if $p=0$, and so we assume that $p>0$ and hence that $\mathscr{A}\neq\emptyset$.

(3.1) *We may assume that if $A\in\mathscr{A}$ and $|A|=m$, then there are at least $(p-2)m+n+1$ members of $\mathscr{A}$ which have non-empty intersection with A.*

Proof. If there are more than $(n-m)(p-2)$ members of $\mathscr{A}$ disjoint from A, then by induction some $p-1$ of them are pairwise disjoint, and these together with A give the required p disjoint members of $\mathscr{A}$. We may assume then that at most $(d-m)(p-2)$ members of $\mathscr{A}$ are disjoint from A; but $|\mathscr{A}|\geqq(p-1)n+1$, and the result follows.

(3.2) $\emptyset\notin\mathscr{A}$.

Proof. This follows from (3.1) by setting $A=\emptyset$.

(3.3) *If* $x\in E$, *then* x *is contained in at most* n *members of* $\mathscr{A}$.

Proof. The sets $A-\{x\}(x\in A\in\mathscr{A})$ are pairwise disjoint subsets of $E-\{x\}$, and at most one of them is empty.

(3.4) *If* $X\subseteqq E$ *and* $|X|\leqq p-1$, *then there exists* $A\in\mathscr{A}$ *with* $A\cap X=\emptyset$.

Proof. By (3.3), the number of members of $\mathscr{A}$ which intersect X is at most $n|X|\leqq\leqq n(p-1)<|\mathscr{A}|$, and the result follows.

Now we define $B_1, \dots, B_p\in\mathscr{A}$ and $x_1, \dots, x_p\in E$, as follows. Inductively, having defined B_j and x_j for $1\leqq j<i$, we let B_i be a member $B\in\mathscr{A}$ of minimum cardinality for which $B\cap\{x_1, \dots, x_{i-1}\}=\emptyset$ (this is possible by (3.4)) and we let x_i be an element $x\in B_i$ for which $|\{A\in\mathscr{A}:x\in A\}|$ is maximum (this is possible by (3.2).) For $1\leqq i\leqq p$, we define

$$\mathscr{A}_i = \{A\in\mathscr{A}: A\cap\{x_1, \dots, x_i\} = \{x_i\}\}$$

$$m_i = |B_i|$$

$$\mathscr{B}_i = \{A\in\mathscr{A}_i\colon |A| = m_i\}$$

$$r_i = |\mathscr{B}_i|$$

$$J_i = \{j:1 \leqq j < i \text{ and some } A\in\mathscr{A}_j \text{ contains } x_i\}.$$

(3.5) $x_1, \dots, x_p$ *are all distinct, and* $m_i\leqq m_j$ *for* $1\leqq i\leqq j\leqq p$.

Proof. These follows immediately from the definitions.

(3.6) *If* $1\leqq i\leqq p$, *then* $r_i\geqq m_i|\mathscr{A}_i|-n+1+\sum_{j\in J_i}(m_j-1)$.

Proof. Clearly $\sum_{A\ni x_i}(|A|-1)\leqq n-1$; but

$$\sum_{A\ni x_i}(|A|-1) = \sum_{A\in\mathscr{A}_i}(|A|-1)+\sum_{1\leqq j\leqq i}\sum_{\substack{A\in\mathscr{A}_j\\ A\ni x_i}}(|A|-1) \geqq$$

$$\geqq \sum_{A\in\mathscr{A}_i} m_i-\sum_{A\in\mathscr{B}_i}1+\sum_{j\in J_i}(m_j-1) =$$

$$= m_i|\mathscr{A}_i|-r_i+\sum_{j\in J_i}(m_j-1).$$

(We observe that if $j\in J_i$, there is a unique $A\in\mathscr{A}_j$ with $x_i\in A$, and of course $|A|\geqq\geqq m_j$.) The result follows.

(3.7) *If* $1\leqq i\leqq p$, *then* $m_i(|\mathscr{A}_i|+|J_i|-p+1)\geqq n$.

Proof. The number of members of $\mathscr{A}$ which intersect B_i is at least $(p-2)m_i+n+1$, by (3.1). However, the number of members of $\mathscr{A}$ which contain $x\in B_i$ is maximized when $x=x_i$, by definition of x_i, and then this number is $|\mathscr{A}_i|+|J_i|$. We deduce that

$$(p-2)m_i+n+1 \leqq m_i(|\mathscr{A}_i|+|J_i|-1)+1$$

and the result follows.

(3.8) *If* $1\leqq i\leqq p$, *then* $r_i\geqq 1+p-i+\sum_{1\leqq j<i}(m_j-1)$.

Proof. From (3.6) and (3.7) we obtain

$$r_i \geqq 1+m_i(p-1-|J_i|)+\sum_{j\in J_i}(m_j-1) =$$

$$= 1+m_i(p-1-|J_i|)-\sum_{\substack{1\leqq j<i\\ j\notin J_i}}(m_j-1)+\sum_{1\leqq j<i}(m_j-1).$$

However, $m_i(p-1-|J_i|)-\sum_{\substack{1\leqq j<i\\ j\notin J_i}}(m_j-1) \geqq m_i(p-1-|J_i|)-\sum_{\substack{1\leqq j<i\\ j\notin J_i}} m_i$ (by (3.5))

$$= m_i(p-1-|J_i|)-m_i(i-1-|J_i|) =$$

$$= m_i(p-i)$$

$$\geqq p-i \quad \text{(by (3.2))}.$$

The result follows.

(3.9) Conclusion. We now construct $A_1, \ldots, A_p\in\mathscr{A}$ as follows. Having defined $A_j (1\leqq j<i)$, we choose $A_i\in\mathscr{B}_i$ with $A_i\cap Z_i=\emptyset$, where Z_i is defined to be

$$\{x_j: i<j\leqq p\}\cup\bigcup_{1\leqq j<i}(A_j-\{x_j\}).$$

To see that this is possible, we observe that the sets $A-\{x_i\} (A\in\mathscr{B}_i)$ are pairwise disjoint, and there are at least $1+|Z_i|$ of them, by (3.8). Thus at least one of them is disjoint from Z_i.

Now for $1\leqq j<i\leqq p$, we have $A_i\cap A_j=\emptyset$. For $x_j\notin A_i$ (since $A_i\in\mathscr{B}_i\subseteq\mathscr{A}_i$), and $A_j-\{x_j\}\subseteq Z_i$ which is disjoint from A_i. Thus $A_1, \ldots, A_p$ are pairwise disjoint, and the theorem is proved.

4. Characterization of equality

(4.1) *If* $\mathscr{A}$ *is a nearly-disjoint collection of subsets of* E, *with* $|E|=n>0$, *and the maximum number of pairwise members of* $\mathscr{A}$ *is* $\frac{1}{n}|\mathscr{A}|$, *then up to isomorphism, one of the following holds:*

(i) $\mathscr{A}=\emptyset$

(ii) $n=1$

(iii) $\mathscr{A}=\{\{1\}, \{1, 2\}, \{1, 3\}, \ldots, \{1, n\}\}$ $(n\geqq 2)$

(iv) $\mathscr{A}=\{\{2, 3, \dots, n\}, \{1, 2,\}, \{1, 3\}, \dots, \{1, n\}\}$ $(n\geqq 3)$
(v) *$\mathscr{A}$ is the set of lines of a projective plane, and E is its set of points*
(vi) *$n\geqq 5$ is odd, and $\mathscr{A}$ contains all two-element subsets of E.*

Proof. Cases (i), (ii) are trivial. We assume then that $\mathscr{A}\neq\emptyset$, and $n\geqq 2$. Put $p-1=\frac{1}{n}|\mathscr{A}|$, so that $p\geqq 2$.

We claim that $\emptyset\notin\mathscr{A}$. For if $\emptyset\in\mathscr{A}$, we have that $p-1\leqq 1+\frac{1}{n}(|\mathscr{A}|-1)$, since at least $\frac{1}{n}(|\mathscr{A}|-1)$ members of $\mathscr{A}-\{\emptyset\}$ are pairwise disjoint; and yet $|\mathscr{A}|=(p-1)n$. This yields that $n=1$, a contradiction.

Suppose that $|A|=1$ for some $A\in\mathscr{A}$, and $A=\{x\}$ say. Now there are at most $(p-2)(n-1)$ members of $\mathscr{A}$ which do not contain x, since at most $p-2$ of them are pairwise disjoint. Thus there are at least

$$(p-1)n-(p-2)(n-1) = n+p-2$$

members of $\mathscr{A}$ which do contain x. But there are certainly no more than n; and so $p=2$, and therefore every member of $\mathscr{A}$ contains x. It follows that we have case (iii).

We may therefore assume that $|A|\geqq 2$ for $A\in\mathscr{A}$. Now if $p=2$, we have either case (iv) or (v), by a result of De Bruijn and Erdős [1]. We assume then that $p\geqq 3$, and proceed by induction on p.

Let us follows through the proof of (1.3), with the hypothesis $|\mathscr{A}|\geqq(p-1)n+1$ replaced by $|\mathscr{A}|=(p-1)n$, and see where it breaks down. In Lemma (3.1) we must replace $(p-2)m+n+1$ by $(p-2)m+n$. (3.2) and (3.3) are trivial, and (3.4) still holds, because in fact the number of members of $\mathscr{A}$ which intersects $X\subseteq E$ is at most $(n-1)|X|<|\mathscr{A}|$. Thus the definition of $B_1, \dots, B_p$ and $x_1, \dots, x_p$ still works. (3.5) is still true, and so is (3.6.) In (3.7) we can only prove that

$$m_i(|\mathscr{A}_i|+|J_i|-p+1) \geqq n-1,$$

but (3.8) is still true except when $i=p$, when we can only prove that $r_p\geqq \sum_{1\leqq j\leqq p}(m_j-1)$. The construction of the conclusion (3.9) still works, except for the construction of A_p, which must fail. We deduce that $r_p=\sum_{1\leqq j\leqq p}(m_j-1)$, and hence that equality holds in the (modified) (3.1) with $A=B_p$ and $m=m_p$. Hence there are exactly $(n-m_p)(p-2)$ members of $\mathscr{A}$ disjoint from B_p, and only $p-2$ of them are pairwise disjoint. Thus by induction, the collection $\{A\in\mathscr{A}: A\cap B_p=\emptyset\}$ is isomorphic to one of type (iv), (v) or (vi). Hence every two-element subset of $E-B_p$ is included in a member of $\mathscr{A}$ disjoint from B_p; and so no $A\in\mathscr{A}$ which intersects B_p has $|A-B_p|>1$. Thus if $A\in\mathscr{A}$ and $A\cap B_p\neq\emptyset$, then either $A=B_p$ or $|A|=2$. In particular, if $x_p\in A\in\mathscr{A}$ and $A\neq B_p$ then $|A|=2$. But $\mathscr{B}_p\neq\{B_p\}$, since

$$r_p = \sum_{1\leqq j<p}(m_j-1) \geqq p-1 \geqq 2,$$

and so there exists $B\in\mathscr{B}_p$ with $B\neq B_p$. Hence $|B|=2$, and so $m_p=2$, and $|B_p|=2$, and so if $x_p\in A\in\mathscr{A}$ then $|A|=2$. However, equality holds in (3.6) when $i=p$, and so

$$\sum_{A\ni x_p}(|A|-1)=n-1.$$

Thus $\{x,x_p\}\in\mathscr{A}$ for every $x\in E-\{x_p\}$. Hence $r_p=n-1-(p-1)$ and so $\sum_{1\leqq j<p}(m_j-1)=n-p$, that is, $p-1=n-p$ (since $2\leqq m_j\leqq m_p=2$ for $1\leqq j<p$). Hence $p=\frac{1}{2}(n+1)$, and so $|\mathscr{A}|=(p-1)n=\frac{1}{2}n(n-1)$. But every $A\in\mathscr{A}$ includes a two-element subset of E, and more than one unless $|A|=2$, and yet no two-element subset is included in more than one member of $\mathscr{A}$. It follows that every member of $\mathscr{A}$ has cardinality 2, and we have case (vi) of the theorem.

5. Remarks

The Erdős—Faber—Lovász conjecture (1.2) also implies the following strengthening of (1.3).

(5.1) (Conjecture) *Let $\mathscr{A}$ be a nearly-disjoint collection of subsets of a set E with $|E|=n>0$. For each $A\in\mathscr{A}$, let $w(A)$ be a nonnegative real number. Then for some $p\geqq 0$, there exist $A_1,\dots,A_p\in\mathscr{A}$, pairwise disjoint, such that*

$$\sum_{1\leqq i\leqq p}w(A_i)\geqq\frac{1}{n}\sum_{A\in\mathscr{A}}w(A)$$

(Thus, our theorem (1.3) yields (5.1) when w is (0,1)-valued.) It would be very interesting to prove (5.1), because one could then apply linear programming duality to obtain a "linear relaxation" of (1.2), the following.

(5.2) (Conjecture) *Let $\mathscr{A}$ be a nearly-disjoint collection of subsets of a set E, with $|E|=n>0$, and let $\mathscr{B}$ be the collection of all sets of members of $\mathscr{A}$ which are pairwise disjoint. Then there is a non-negative real-valued function q on $\mathscr{B}$ such that*

(i) $\sum_{B\in\mathscr{B}}q(B)=n$

(ii) *for each* $A\in\mathscr{A}$, $\sum_{A\in B\in\mathscr{B}}q(B)\leqq 1$.

(We can choose such a q integer-valued if and only if (1.2) is true.) It is tantalizing that our proof of (1.3) nearly serves to prove (5.1). For example, if $\mathscr{A}$ is a minimum counterexample to (5.1), and $B\in\mathscr{B}$ has $\frac{|B|}{w(B)}$ minimum, it is easy to see that a large number of the members of $\mathscr{A}$ intersecting B have this ratio almost as small. (This is analogous to (3.8).)

(1.2) is of particular interest when all members of $\mathscr{A}$ have the same cardinality, because of its connection with block designs. However, in that case (1.3) seems to be nowhere near best possible. For example, if $\mathscr{A}$ is a Steiner triple system, then $|\mathscr{A}|=\frac{1}{6}n(n-1)$, and (1.3) guarantees $\frac{1}{6}(n-1)$ disjoint triples. However, Woolbright [3] has shown that there are in fact at least $\frac{3}{10}n-7$ disjoint triples.

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P. D. Seymour

Department of Mathematics
The Ohio State University
Columbus, Ohio 43210, USA

Note added in proof. Hindman [4] has shown that the conjecture (1.1) is true for $n\leqq 10$, and he also proves that (1.1) and (1.2) are equivalent.

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